# Time-to-Go Prediction for Homing Missiles Based on Minimum-Time Intercepts

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**Two minimum-time trajectories are developed for use as a time-to-go algorithm for a missile controlled by a linear-quadratic guidance law. The first algorithm is based on the minimum normal-acceleration-weighted final time trajectory for a constant speed missile, and the second algorithm is based on the minimum final time trajectory of an accelerating missile with bounded normal acceleration. Because the initial off-boresight angle is fixed, these algorithms are expected to work well for intercepts where large off-boresight angles occur. Both time-to-go algorithms are tested in a six-degree-of-freedom simulation of a bank-to-turn missile attacking a smart target, and their miss distances are compared with those obtained by time-to-go algorithms known as range over closing speed and accelerated range over closing speed. Both algorithms give low miss distances for a wide range of intercept geometries including those with large off-boresight angles. These results are substantially improved relative to those obtained with range over closing speed. However, the accelerated range over closing speed algorithm matches the minimum-time results and is the preferred algorithm because of its simplicity.**

## Nomenclature

$A,B$ = functions defined in Eqs. (47) and (49)
$a,b,c$ = constants defined in Eqs. (21–23)
$a_n$ = missile normal acceleration, ft/s$^2$
$a_t$ = missile tangential acceleration, ft/s$^2$
$a_1,b_1$ = constants defining $V_M$ for thrusting flight
$a_2,b_2$ = constants defining $V_M$ for coasting flight
$d$ = miss distance, ft
$K$ = modulus of elliptic functions
$k$ = thrusting, $k=1$; coasting, $k=2$
$P$ = magnitude of missile acceleration oscillation
$t$ = time, s
$V_M$ = missile velocity, ft/s
$V_T$ = target velocity, ft/s
$v$ = ratio of target speed to missile speed
$W$ = performance index weight
$X,Y$ = inertial coordinates, ft
$x,y$ = relative coordinates, ft
$\alpha_n$ = nondimensional missile normal acceleration
$\gamma$ = angle of maximum normal acceleration
$\xi,\eta$ = nondimensional relative coordinates
$\theta$ = orientation of missile velocity vector to $X$ axis
$\tau$ = nondimensional time
$\phi$ = orientation of target velocity vector to $X$ axis
$\psi$ = transformed variable representing $\theta$

## I. Introduction

ALTHOUGH most existing missiles are guided by proportional navigation, a linear-quadratic guidance law is being considered for future bank-to-turn missile designs.[1,2] Although the linear-quadratic guidance rule contains proportional navigation as a particular case for small line-of-sight angles and negligible acceleration along the line of sight,[3] most envisioned missile engagements exceed these limits because of high tangential and normal accelerations. If the purpose of the guidance law is to enhance state estimation as well as to achieve a small miss distance, the linear-quadratic guidance law proposed in Ref. 2 can be used. The effect of this guidance law is to keep the line of sight in motion and can lead to a large off-boresight angle (angle between missile longitudinal axis and line-of-sight) at the beginning of the intercept trajectory. A characteristic of both linear-quadratic guidance laws is that an estimate of time-to-go is needed to calculate the gains of the control law.

A commonly used time-to-go formula is range over closing speed. For this predictor to be accurate, the missile and target must be moving at constant speed in the intercept geometry. An improved time-to-go estimator[4] uses a simple model of missile tangential acceleration to produce a more accurate value than that of range over closing speed. Here, it is assumed that the missile and the constant speed target move along straight lines in the intercept geometry. Other time-to-go algorithms have been proposed in Refs. 5 and 6. In Ref. 5, a variable-speed missile in rectilinear motion intercepts a target whose acceleration is constant in inertial space. In Ref. 6, the flight time of a missile with a given initial acceleration profile ($X$ inertial acceleration is either zero or constant) intercepting a zero-acceleration target and minimizing a quadratic perfor-

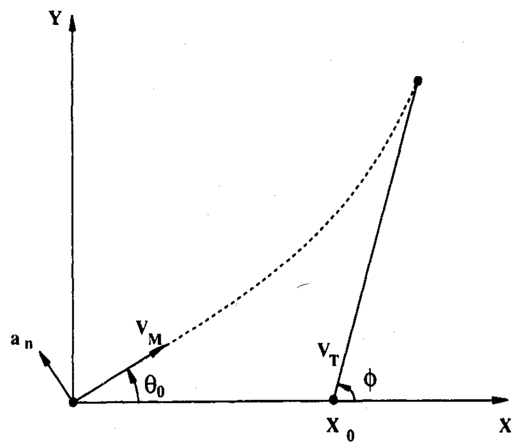


Fig. 1 Launch geometry.

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mance index, which is the weighted sum of the miss distance squared and the final time squared, is used as the time-to-go.

The purpose of this paper is to find time-to-go algorithms in two dimensions using models that are more representative of the actual physics. Hence, two minimum-time intercept trajectories are developed for a nonaccelerating target assuming that the normal acceleration of the missile can be controlled within limits. First, in Sec. II, the missile is assumed to have constant speed, and the normal acceleration limit is imposed in the form of a penalty on the final time. Second, in Sec. III, the missile is assumed to have a known tangential acceleration profile (constant acceleration while thrusting followed by constant deceleration while coasting), and the normal acceleration limits are imposed by direct bounds. The minimum final time can then be used as the time-to-go estimate for the linear-quadratic guidance law.

In Sec. IV, both minimum-time algorithms are used in a simulation in which an accelerating missile with a linear-quadratic guidance law attacks an accelerating target. These results are compared with those of using range over closing speed without and with missile acceleration.

## II. Minimum Weighted-Final-Time Trajectories

The launch geometry of the intercept problem is shown in Fig. 1. The $XY$ coordinate system represents an inertial frame, and the $X$ axis is along the line of sight at $t=0$. The target, located at $X_T=X_0$ at $t=0$, is moving along a straight line that makes an angle $\phi$ with respect to the $X$ axis. The constant-speed missile is launched at an angle $\theta_0$ relative to the $X$ axis, and the velocity direction $\theta(t)$ is changed by controlling the normal acceleration $a_n(t)$.

In terms of the nondimensional quantities

$$\tau = t(V_M/X_0), \qquad \xi = \left[(X_T - X_M)/X_0\right]$$

$$\eta = \left[(Y_T - Y_M)/X_0\right], \qquad \alpha_n = \left[(a_n X_0)/V_M^2\right]$$

$$\nu = (V_T/V_M) \tag{1}$$

the optimal intercept problem is stated as follows: Find the normal acceleration history $\alpha_n(\tau)$ that minimizes the performance index

$$J = W(\tau_f - \tau_0) + \frac{1-W}{2}\int_{\tau_0}^{\tau_f} \alpha_n^2 \, d\tau \tag{2}$$

subject to the differential constraints

$$\dot{\xi} = \nu \cos\phi - \cos\theta \tag{3}$$

$$\dot{\eta} = \nu \sin\phi - \sin\theta \tag{4}$$

$$\dot{\theta} = \alpha_n \tag{5}$$

and the prescribed boundary conditions

$$\tau_0 = 0, \qquad \xi_0 = 1, \qquad \eta_0 = 0, \qquad \theta_0 = \theta_{0_s} \tag{6}$$

$$\xi_f = 0, \qquad \eta_f = 0 \tag{7}$$

where $\theta_{0_s}$ is the specified value of $\theta_0$.

The performance index is the weighted sum of the final time and the integral of the normal acceleration squared. Without the integral, the solution of the optimal control problem is the infinite normal acceleration to rotate the missile velocity to the homing triangle direction followed by zero normal acceleration until intercept. By adjusting the weight $W$, the highest normal acceleration encountered along the trajectory can be controlled.

### A. Solution of the Optimal Control Problem

From the variational Hamiltonian

$$H = \frac{1-W}{2}\alpha_n^2 + \lambda_\xi(\nu\cos\phi - \cos\theta) + \lambda_\eta(\nu\sin\phi - \sin\theta) + \lambda_\theta(\alpha_n) \tag{8}$$

it is seen that the Euler-Lagrange equations are given by

$$\dot{\lambda}_\xi = -H_\xi = 0 \tag{9}$$

$$\dot{\lambda}_\eta = -H_\eta = 0 \tag{10}$$

$$\dot{\lambda}_\theta = -H_\theta = -\lambda_\xi \sin\theta + \lambda_\eta \cos\theta \tag{11}$$

$$0 = H_{\alpha_n} = (1-W)\alpha_n + \lambda_\theta \tag{12}$$

Equations (9) and (10) state that $\lambda_\xi$ and $\lambda_\eta$ must be constants. Before solving the remaining equations, the natural boundary conditions are derived.

From the end-point function

$$G = W(\tau_f - \tau_0) + \nu_\xi \xi_f + \nu_\eta \eta_f \tag{13}$$

it is seen that the natural boundary conditions are

$$H_f = -G_{\tau_f} = -W \tag{14}$$

$$\lambda_{\xi f} = G_{\xi_f} = \nu_\xi \tag{15}$$

$$\lambda_{\eta f} = G_{\eta_f} = \nu_\eta \tag{16}$$

$$\lambda_{\theta f} = G_{\theta_f} = 0 \tag{17}$$

Equations (12) and (17) imply that

$$\alpha_{nf} = 0 \tag{18}$$

Since the Hamiltonian is not an explicit function of time, the first integral $H=\text{const}$ exists. When combined with Eq. (14), the first integral becomes

$$H = -W \tag{19}$$

Hence, if the definition of Eq. (8) for $H$ is substituted into Eq. (19), the following expression for the optimal control can be obtained

$$\alpha_n = \pm\sqrt{\left[2/(1-W)\right](a + b\sin\theta + c\cos\theta)} \tag{20}$$

where

$$a = W + \lambda_\xi \nu \cos\phi + \lambda_\eta \nu \sin\phi \tag{21}$$

$$b = -\lambda_\eta \tag{22}$$

$$c = -\lambda_\xi \tag{23}$$

If $\theta_0$ were free, the optimal $\alpha_n$ would be $\alpha_n = 0$, and the optimal intercept path would be $\theta = \text{const} \equiv \theta_{SL}$ or a straight line. Hence, if $\theta_0 < \theta_{SL}$, the plus sign in Eq. (20) holds, and if $\theta_0 > \theta_{SL}$, the minus sign is valid.

Equation (20) can also be obtained by integrating Eq. (11). This is accomplished by changing the independent variable from $\tau$ to $\theta$ using Eq. (5), using Eq. (12) to relate $\alpha_n$ to $\lambda_\theta$, integrating, and applying the final condition Eq. (17). Finally, $\theta_f$ is eliminated by using the final condition Eq. (14).

To integrate the differential Eqs. (3-5), it is useful to rewrite Eq. (20) in the elliptic function form[7]:

$$\alpha_n = \pm\sqrt{\left[2/(1-W)\right](a+P)(1-K^2\sin^2\psi)} \tag{24}$$

where

$$\psi = \frac{\theta - \gamma}{2}, \quad \tan\gamma = \frac{b}{c}, \quad P = \sqrt{B^2 + C^2}, \quad K^2 = \frac{2P}{a + P} \tag{25}$$

If $\tau$ is chosen to be the dependent variable, Eq. (5) can be integrated as

$$\tau_{go} = \tau_f - \tau_0 = \int_{\theta_0}^{\theta_f} \frac{d\theta}{\alpha_n(\theta)} = 2\int_{\psi_0}^{\psi_f} \frac{d\psi}{\alpha_n(\psi)} \tag{26}$$

Then, substitution of Eq. (24) leads to

$$\tau_{go} = \pm K\sqrt{[(1-W)/P]}\,[F(\psi_f, K) - F(\psi_0, K)] \tag{27}$$

where $F(\psi, K)$ is the elliptic integral of the first kind.

Similarly, Eqs. (3) and (4) can be expressed as

$$\xi - \xi_0 = 2\int_{\psi_0}^{\psi} \frac{[\nu\cos\phi - \cos(2\psi + \gamma)]\,d\psi}{\alpha_n(\psi)} \tag{28}$$

$$\eta - \eta_0 = 2\int_{\psi_0}^{\psi} \frac{[\nu\sin\phi - \sin(2\psi + \gamma)]\,d\psi}{\alpha_n(\psi)} \tag{29}$$

Carrying out the integration and applying the initial conditions of Eq. (6) leads to

$$\begin{aligned} \xi = \pm K\sqrt{[(1-W)/P]}\Big\{&\nu\cos\phi - [1 - (2/K^2)]\cos\gamma[F(\psi, K) \\ &- F(\psi_0, K)] - (2\cos\gamma/K^2)[E(\psi, K) - E(\psi_0, K)] \\ &- (\sin\gamma/K^2)\big[\sqrt{1 - K^2\sin^2\psi} - \sqrt{1 - K^2\sin^2\psi_0}\big]\Big\} + 1 \end{aligned} \tag{30}$$

and

$$\begin{aligned} \eta = \pm K\sqrt{[(1-W)/P]}\Big\{&\big[\nu\sin\phi - [1 - (2/K^2)]\sin\gamma\big] \\ &\times [F(\psi, K) - F(\psi_0, K)] - (2\sin\gamma/K^2)[E(\psi, K) - E(\psi_0, K)] \\ &+ (\cos\gamma/K^2)\big[\sqrt{1 - K^2\sin^2\psi} - \sqrt{1 - K^2\sin^2\psi_0}\big]\Big\} \end{aligned} \tag{31}$$

where $E(\psi, K)$ is the elliptic integral of the second kind.

The last step is to satisfy the final conditions of Eq. (7). This amounts to finding the values of $\lambda_\xi$ and $\lambda_\eta$ which satisfy Eqs. (30) and (31) evaluated at $\psi_f$, where $\xi_f = \eta_f = 0$. Because these equations are nonlinear, the solution process is iterative. Rather than solving for the Lagrange multipliers, it is more convenient to work in terms of $\psi_0$ and $\psi_f$. The relationships between the $\psi$ and the $\lambda$ are obtained from Eq. (24) and are given by

$$\lambda_\xi = \frac{-W\cos(\theta_0 - 2\psi_0)}{2\sin^2\psi_f - 1 + \nu\cos(\phi - \theta_0 + 2\psi_0)} \tag{32}$$

$$\lambda_\eta = \frac{-W\sin(\theta_0 - 2\psi_0)}{2\sin^2\psi_f - 1 + \nu\cos(\phi - \theta_0 + 2\psi_0)} \tag{33}$$

### B. Numerical Results

Given values for $W$, $\nu$, $\phi$, and $\theta_0$, the values of $\psi_0$ and $\psi_f$ which satisfy Eqs. (30) and (31) evaluated at the final point, Eq. (7), are computed using Newton's method with analytical derivatives. Given $\psi_0$ and $\psi_f$, $\lambda_\xi$ and $\lambda_\eta$ follow from Eqs. (32) and (33). Next, Eqs. (21-23) give $A$, $B$, and $C$, and Eqs. (25) give $\gamma$, $P$, and $K$. Finally, the values of $\xi_f$ and $\eta_f$ can be computed from Eqs. (30) and (31). Improved values for $\psi_0$ and $\psi_f$ (values which make $\xi_f$ and $\eta_f$ closer to zero) are provided by Newton's method. For $\phi = 0$ deg and $0 \le \nu \le 1.0$, the following ranges of values for $\psi_0$ and $\psi_f$ have been computed:

$$-70 \text{ deg} \le \theta_0 \le 0 \text{ deg}, \qquad 0 \text{ deg} \le \psi_0 \le 90 \text{ deg}$$

$$0 \text{ deg} \le \theta_0 \le 70 \text{ deg}, \qquad -90 \text{ deg} \le \psi_0 \le 0 \text{ deg}$$

$$-70 \text{ deg} \le \theta_0 \le 0 \text{ deg}, \qquad 50 \text{ deg} \le \psi_f \le 90 \text{ deg}$$

$$0 \text{ deg} \le \theta_0 \le 70 \text{ deg}, \qquad -90 \text{ deg} \le \psi_f \le -50 \text{ deg}$$

The range of values used for $\theta_0$ correspond to a typical seeker field of view. Note that there is a discontinuity in $\psi_0$ and $\psi_f$ at $\theta_0 = 0$ from $+90$ deg to $-90$ deg. Additional results are presented in Ref. 8.

## III. Minimum-Time Trajectories with Bounded Control

In the previous formulation, it has been necessary to assume $V_M$ as a constant in order to get an analytical solution. Here, it is possible to include the tangential acceleration of the missile. If $x = X_T - X_M$ and $y = Y_T - Y_M$, the equations of motion of the engagement in relative coordinates are given by

$$\dot{x} = V_T\cos\phi - V_M\cos\theta \tag{34}$$

$$\dot{y} = V_T\sin\phi - V_M\sin\theta \tag{35}$$

$$\dot{\theta} = a_n / V_M \tag{36}$$

The tangential acceleration history of the missile is assumed to be constant $a_{t_{max}} > 0$ while thrusting, that is, for $t \le t_c$ where $t_c$ is the known engine cutoff time, and a constant $a_{t_{min}} < 0$ while coasting. As a consequence, the velocity of the missile while thrusting is given by

$$V_M = a_1 + b_1 t, \qquad t \le t_c \tag{37}$$

where

$$a_1 = V_{M_0}, \qquad b_1 = a_{t_{max}} \tag{38}$$

Similarly, the velocity of the missile during the coasting phase is given by

$$V_M = a_2 + b_2 t, \qquad t \ge t_c \tag{39}$$

where

$$a_2 = V_{M_0} + (a_{t_{max}} - a_{t_{min}})t_c, \qquad b_2 = a_{t_{min}} \tag{40}$$

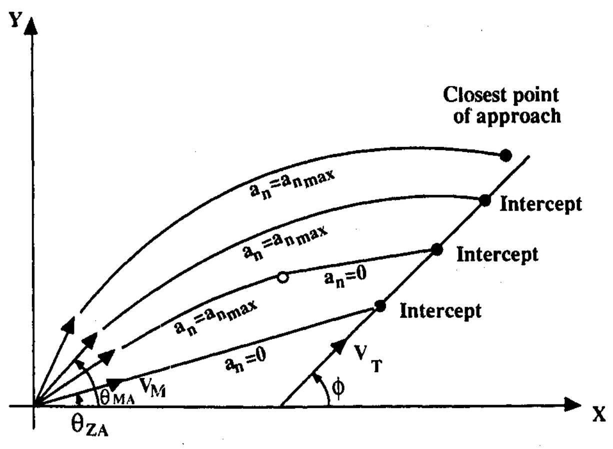


**Fig. 2 Typical trajectories for bounded $a_n$.**

The prescribed initial conditions are

$$t_0 = t_{0_s}, \qquad x_0 = x_{0_s}, \qquad y_0 = y_{0_s}, \qquad \theta_0 = \theta_{0_s} \tag{41}$$

where the subscript $s$ denotes a specific value. Intercept at the final point requires that

$$x_f = 0, \qquad y_f = 0 \tag{42}$$

If $\theta_0$ were free, the control for the minimum-time trajectory would be $a_n = 0$, that is, a straight line. On the other hand, if $\theta_0$ were prescribed, the minimum-time control would be infinite normal acceleration to rotate the velocity vector instantaneously to the above straight line followed by zero normal acceleration. If a bound were applied to the normal acceleration, the optimal control would become maximum normal acceleration followed by zero normal acceleration, that is, a bang-bang control. The bang-bang control is used here to generate a minimum-time trajectory and the flight time is used as a prediction of the time-to-go.

In the solution of the problem, it is found that a direct intercept can be achieved for $\theta_{ZA} \le \theta_0 \le \theta_{MA}$ (see Fig. 2), where $\theta_{ZA}$ is the initial angle for the zero-normal-acceleration intercept and $\theta_{MA}$ is the initial angle for the maximum-normal-acceleration intercept. For $\theta_0 > \theta_{MA}$, the missile passes in front of the target during the maximum normal acceleration phase. Then, it performs a 360-deg turn before it goes for the intercept. When this happens, the minimum time is taken as the time to the point of closest approach. A similar discussion holds for $\theta_0 < \theta_{ZA}$.

In the development of the equations, there are four important times: the initial time $t_0$, the final time $t_f$, the engine cutoff time $t_c$, and the switch time $t_s$ between $a_n = a_{n_{\max}}$ and $a_n = 0$. It is assumed that

$$t_0 \le t_s \le t_f \tag{43}$$

so that $a_n = a_{n_{\max}}$ all of the way if $t_s = t_f$, and $a_n = 0$ all of the way if $t_s = t_0$. In the development of the equations, the engine cutoff time is assumed to satisfy the inequality

$$t_0 \le t_c \le t_f \tag{44}$$

Then, if $t_c < t_0$ (coasting all of the way), the correct equations can be obtained by setting $t_c = t_0$, and if $t_c > t_f$ (thrusting all of the way), $t_c$ is set equal to $t_f$.

### A. Constant Normal Acceleration

The equations of motion, Eqs. (34–36), can be integrated for the case where $a_n$ is constant. These solutions are valid for the cases where $a_n = a_{n_{\max}}$ or $a_n = 0$.

Since the missile velocity has the general form $V_M = a_k + b_k t$, where $k = 1$ for thrusting and $k = 2$ for coasting, Eq. (36) can be integrated as

$$\theta = \theta_p + \frac{a_n}{b_k} \ln\left(\frac{a_k + b_k t}{a_k + b_k t_p}\right) \tag{45}$$

The subscript $p$ denotes a generic starting point; it could be the initial point, the engine cutoff point, or the switch point.

Next, with Eq. (45), Eq. (34) can be integrated to yield

$$x = x_p + A(t, a_n, k, t_p, \theta_p) - A(t_p, a_n, k, t_p, \theta_p) \tag{46}$$

where

$$A(t,a_n,k,t_p,\theta_p) = V_T \cos\phi t - \frac{2b_k(a_k + b_k t)^2}{4b_k^2 + a_n^2} \times \left\{ \cos\left[\theta_p + \frac{a_n}{b_k}\ln\left(\frac{a_k + b_k t}{a_k + b_k t_p}\right)\right] + \frac{a_n}{2b_k}\sin\left[\theta_p + \frac{a_n}{2b_k}\ln\left(\frac{a_k + b_k t}{a_k + b_k t_p}\right)\right]\right\} \tag{47}$$

Integration by parts is used to obtain the second term in Eq. (47). Finally, Eq. (35) for $y$ is integrated in the same way as Eq. (34) and leads to

$$y = y_p + B(t, a_n, k, t_p, \theta_p) - B(t_p, a_n, k, t_p, \theta_p) \tag{48}$$

where

$$B(t,a_n,k,t_p,\theta_p) = V_T \sin\phi t - \frac{2b_k(a_k + b_k t)^2}{4b_k^2 + a_n^2} \times \left\{ \sin\left[\theta_p + \frac{a_n}{b_k}\ln\left(\frac{a_k + b_k t}{a_k + b_k t_p}\right)\right] - \frac{a_n}{2b_k}\cos\left[\theta_p + \frac{a_n}{b_k}\ln\left(\frac{a_k + b_k t}{a_k + b_k t_p}\right)\right]\right\} \tag{49}$$

### B. Zero-Normal-Acceleration Intercept

In this section, the zero-normal-acceleration intercept (straight-line intercept $\theta = \theta_{ZA}$) is derived. It is the dividing line between the $a_n \ge 0$ ($\theta_0 \le \theta_{ZA}$) intercepts and $a_n \le 0$ ($\theta_0 \ge \theta_{ZA}$) intercepts. For the time being, the cutoff time, which is known, is assumed to satisfy the inequality constraint $t_0 < t_c < t_f$.

If $a_n = 0$, Eqs. (45), (46), and (48) can be applied at $t_c$ and $t_f$ and the results combined to yield

$$\theta_f = \theta_0 \tag{50}$$

$$x_f = x_0 + A(t_c,0,1,t_0,\theta_0) - A(t_0,0,1,t_0,\theta_0) + A(t_f,0,2,t_c,\theta_0) - A(t_f,0,2,t_c,\theta_0) \tag{51}$$

$$y_f = y_0 + B(t_c,0,1,t_0,\theta_0) - B(t_0,0,1,t_0,\theta_0) + B(t_f,0,2,t_c,\theta_0) - B(t_f,0,2,t_c,\theta_0) \tag{52}$$

For an intercept, $x_f = y_f = 0$ so that Eqs. (50–52) involve two unknowns: $t_f$ and $\theta_0 \equiv \theta_{ZA}$. These equations can be solved for $\cos\theta_{ZA}$ and $\sin\theta_{ZA}$ as

$$\cos\theta_{ZA} = (V_T \cos\phi t_f + x_0)/D \qquad \sin\theta_{ZA} = (V_T \sin\phi t_f + y_0)/D \tag{53}$$

which, in turn, can be squared and added to obtain the single equation for $t_f$:

$$(V_T \cos\phi t_f + x_0)^2 + (V_T \sin\phi t_f + y_0)^2 = D^2 \tag{54}$$

Once $t_f$ is known, $\theta_{ZA}$ follows from Eqs. (53). If $t_c \ge t_f$, the proper equations can be obtained from Eqs. (53) and (54) by setting $t_c = t_f$. On the other hand, for $t_c \le t_0$, set $t_c = t_0$.

### C. Maximum-Normal-Acceleration Intercept

The maximum-normal-acceleration intercept is the dividing line between intercept trajectories and closest-point-of-approach trajectories. It is possible to determine this trajectory by applying Eqs. (45), (46), and (48). However, a system of two equations in two unknowns results, and while it can be solved efficiently with careful coding, a simpler approach is used to determine whether an intercept is possible.

Once $\theta_{ZA}$ is known, the actual $\theta_0$ can be compared with $\theta_{ZA}$ to determine the sign of $a_n$: $\theta_0 \le \theta_{ZA}$, $a_n \ge 0$ or $\theta_0 \ge \theta_{ZA}$, $a_n \le 0$. Then, the equations of motion are integrated by Euler integration for $a_n = a_{n_{\max}}$ over the entire trajectory. For the case where $a_n < 0$, the value of $y_f$ at $x_f = 0$ is investigated to determine whether an intercept can occur. If the missile passes behind the target ($y_f > 0$), an intercept can be achieved, and if the missile passes ahead of the target ($y_f < 0$), no intercept is possible. If the missile turn has a small enough radius so that $x_f \neq 0$ after a reasonable time, an intercept is deemed possible.

**D. Intercept Trajectories**

For an intercept, the switch time is assumed to satisfy the inequality $t_0 < t_s < t_f$ because $t_s = t_0$ is the zero-normal-acceleration intercept, and $t_s = t_f$ is the maximum-normal-acceleration intercept. There are four possible configurations for an intercept trajectory depending on the value of the known engine cutoff time; that is $t_c \le t_0$, $t_0 \le t_c \le t_s$, $t_s \le t_c \le t_f$, and $t_c \ge t_f$. The equations for $t_c \le t_0$ can be obtained from those for $t_0 \le t_c \le t_s$ by setting $t_c = t_0$, and the equations for $t_c \ge t_f$ can be obtained from those for $t_s \le t_c \le t_f$ by setting $t_c = t_f$. Hence, only two sets of equations need to be derived.

*1.* $t_0 \le t_c \le t_s$

For this case, Eqs. (45), (46), and (48) can be applied at $t_c$ and $t_s$ and combined to yield

$$\theta_s = \theta_c + \frac{a_n}{b_2}\ln\left(\frac{a_2+b_2t_s}{a_2+b_2t_c}\right), \quad \theta_c = \theta_0 + \frac{a_n}{b_1}\ln\left(\frac{a_1+b_1t_c}{a_1+b_1t_0}\right) \tag{55}$$

$$x_s = x_0 + A(t_c,a_n,1,t_0,\theta_0) - A(t_0,a_n,1,t_0,\theta_0) + A(t_s,a_n,2,t_c,\theta_c) - A(t_c,a_n,2,t_c,\theta_c) \tag{56}$$

$$y_s = y_0 + B(t_c,a_n,1,t_0,\theta_0) - B(t_0,a_n,1,t_0,\theta_0) + B(t_s,a_n,2,t_c,\theta_c) - B(t_c,a_n,2,t_c,\theta_c) \tag{57}$$

Along the straight-line part ($a_n = 0$), Eqs. (45), (46), and (48) become

$$\theta_f = \theta_s \tag{58}$$

$$x_f = x_s + A(t_f,0,2,t_s,\theta_s) - A(t_s,0,2,t_s,\theta_s) \tag{59}$$

$$y_f = y_s + B(t_f,0,1,t_s,\theta_s) - B(t_s,0,2,t_s,\theta_s) \tag{60}$$

Because of Eqs. (47) and (49), the explicit forms of Eqs. (59) and (60) are given by

$$x_f = x_s + \left\{V_T\cos\phi - \left[a_2 + ½ b_2(t_f - t_s)\right]\cos\theta_s\right\}(t_f - t_s) \tag{61}$$

$$y_f = y_s + \left\{V_T\sin\phi - \left[a_2 + ½ b_2(t_f - t_s)\right]\sin\theta_s\right\}(t_f - t_s) \tag{62}$$

The solution process is to vary $t_s$ until $x_f$ and $y_f$ are both zero. This is accomplished by setting $x_f = 0$ and solving Eq. (61) analytically for $t_f$. Equation (60) is a quadratic equation and has two roots. The correct root is the smallest value of $t_f$ which is larger than $t_s$. Then, the solution for $t_f$ is substituted into Eq. (62) where, because $t_s$ is guessed, $y_f \ne 0$. Hence, $t_s$ is varied until $y_f = 0$. In this way, it is only necessary to solve one equation in one unknown.

*2.* $t_s \le t_c \le t_f$

If the engine cutoff time is between the switch time and the final time, Eqs. (45–47) lead to

$$\theta_c = \theta_s, \qquad \theta_s = \theta_0 + \frac{a_n}{b_1}\ln\left(\frac{a_1+b_1t_s}{a_1+b_1t_0}\right) \tag{63}$$

$$x_c = x_0 + A(t_s,a_n,1,t_0,\theta_0) - A(t_0,a_n,1,t_0,\theta_0) + A(t_c,0,1,t_s,\theta_s) - A(t_s,0,1,t_s,\theta_s) \tag{64}$$

$$y_c = y_0 + B(t_s,a_n,1,t_0,\theta_0) - B(t_0,a_n,1,t_0,\theta_0) + B(t_c,0,1,t_s,\theta_s) - B(t_s,0,1,t_s,\theta_s) \tag{65}$$

For the coast, the corresponding equations are

$$\theta_f = \theta_c \tag{66}$$

$$x_f = x_c + A(t_f,0,2,t_c,\theta_c) - A(t_c,0,2,t_c,\theta_c) \tag{67}$$

$$y_f = y_c + B(t_f,0,2,t_c,\theta_c) - B(t_c,0,2,t_c,\theta_c) \tag{68}$$

The explicit forms of Eqs. (67) and (68) are similar to those of Eqs. (61) and (62). Hence, the same solution process can be used.

**E. Closest-Point-of-Approach Trajectories**

For $a_n = a_{n_{\max}}$ over the entire trajectory and for $t_0 \le t_c \le t_f$, Eqs. (45), (46), and (48) can be applied at $t_c$ and $t_f$ and combined to obtain

$$\theta_f = \theta_c + \frac{a_n}{b_2}\ln\left(\frac{a_2+b_2t_f}{a_2+b_2t_c}\right), \quad \theta_c = \theta_0 + \frac{a_n}{b_1}\ln\left(\frac{a_1+b_1t_c}{a_1+b_1t_0}\right) \tag{69}$$

$$x_f = x_0 + A(t_c,a_n,1,t_0,\theta_0) - A(t_0,a_n,1,t_0,\theta_0) + A(t_f,a_n,2,t_c,\theta_c) - A(t_c,a_n,2,t_c,\theta_c) \tag{70}$$

$$y_f = y_0 + B(t_c,a_n,1,t_0,\theta_0) - B(t_0,a_n,1,t_0,\theta_0) + B(t_f,a_n,2,t_c,\theta_c) - B(t_c,a_n,2,t_c,\theta_c) \tag{71}$$

The equations for the case where $t_c < t_0$ are obtained from Eqs. (69–71) by setting $t_c = t_0$. Similarly, for $t_c > t_f$, set $t_c = t_f$.

The point of closest approach is obtained by minimizing the performance index

$$d^2 = x_f^2 + y_f^2 \tag{72}$$

This is accomplished by solving the algebraic equation

$$\frac{\partial d^2}{\partial t_f} = 0 \tag{73}$$

by bisection. The derivative in Eq. (73) can be taken analytically. To verify that the solution is a minimum, the second derivative $\partial^2 d^2/\partial t_f^2$ is checked.

**F. Numerical Results**

The algorithm used to compute a trajectory is the following:

1) Given $V_T$, $\phi$, $V_{M0}$, $a_{t_{\max}}$, $a_{t_{\min}}$, $t_0$, $x_0$, $y_0$, and $t_c$, compute $\theta_{ZA}$. This determines whether $a_n > 0$ or $a_n < 0$.

2) Given $a_{n_{\max}}$ and $\theta_0$, determine whether or not an intercept can occur.

3) If an intercept is possible, calculate the switch time for $x_f = y_f = 0$.

4) If an intercept is not possible, calculate the final time for closest point of approach.

Some numerical results are contained in Ref. 9 for different values of the design parameters that are typical of a conceptual bank-to-turn missile.

## IV. Simulation Results

The two methods for predicting time-to-go have been tested in a deterministic six-degree-of-freedom simulation of a bank-to-turn homing missile with a linear-quadratic guidance law[1,2] for which the gains vary inversely with the time-to-go. At launch, the missile and the target are 10,000 ft altitude and 0.9 Mach number, with the target flying at constant altitude in a straight line. At launch, the orientation of the target relative to the missile is defined by the range $X_0$ (7000 ft or 3000 ft), the aspect angle $\phi_0$ (0–180 deg) which is the angle between the target longitudinal axis and the line of sight, and the off-boresight angle $\theta_0$ (0–40 deg) which is the angle between the line of sight and the missile longitudinal axis.

During flight, the missile is capable of pulling 100 $g$ normal acceleration, while its tangential acceleration profile can be approximated by $a_{t_{\max}} = 25\ g$ while thrusting (burn

**Table 1 Simulation results**

| $n$ | $\theta_0$ | $X_0$ | $\phi_0$ | $d_{\mathrm{I}}$ | $d_{\mathrm{II}}$ | $d_{\mathrm{III}}$ | $d_{\mathrm{IV}}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 7000 | 0 | 0.156 | 0.331 | 0.337 | 0.219 |
| 2 | 0 | 7000 | 30 | 0.695 | 0.243 | 0.243 | 0.266 |
| 3 | 0 | 7000 | 60 | 1.205 | 0.282 | 0.284 | 0.532 |
| 4 | 0 | 7000 | 90 | 1.318 | 0.473 | 0.469 | 0.823 |
| 5 | 0 | 7000 | 120 | 2.560 | 0.258 | 0.265 | 0.342 |
| 6 | 0 | 7000 | 150 | 0.734 | 0.073 | 0.077 | 3.199 |
| 7 | 0 | 7000 | 180 | 0.352 | 0.009 | 0.010 | 1.521 |
| 8 | 0 | 3000 | 0 | 0.995 | 0.096 | 0.097 | 0.218 |
| 9 | 0 | 3000 | 30 | 0.685 | 0.072 | 0.069 | 0.306 |
| 10 | 0 | 3000 | 60 | 1.401 | 0.065 | 0.063 | 1.497 |
| 11 | 0 | 3000 | 90 | 2.524 | 0.165 | 0.138 | 2.563 |
| 12 | 0 | 3000 | 120 | 13.52 | 1.073 | 2.950 | 5.354 |
| 13 | 0 | 3000 | 150 | 4.219 | 4.641 | 2.911 | 11.82 |
| 14 | 0 | 3000 | 180 | 0.585 | 0.260 | 0.253 | 4.562 |
| 15 | 40 | 7000 | 0 | 359.2 | 1055 | 1251 | 1159 |
| 16 | 40 | 7000 | 30 | 259.7 | 1066 | 1450 | 1112 |
| 17 | 40 | 7000 | 60 | 0.164 | 0.704 | 0.970 | 0.636 |
| 18 | 40 | 7000 | 90 | 0.922 | 0.329 | 0.296 | 0.532 |
| 19 | 40 | 7000 | 120 | 1.944 | 0.130 | 0.092 | 0.758 |
| 20 | 40 | 7000 | 150 | 1.016 | 0.305 | 0.323 | 0.672 |
| 21 | 40 | 7000 | 180 | 0.358 | 0.058 | 0.063 | 1.609 |
| 22 | 40 | 3000 | 0 | 2104 | 0.167 | 0.041 | 4.084 |
| 23 | 40 | 3000 | 30 | 2695 | 0.254 | 0.238 | 0.640 |
| 24 | 40 | 3000 | 60 | 2809 | 0.173 | 0.165 | 0.198 |
| 25 | 40 | 3000 | 90 | 2519 | 0.163 | 0.267 | 0.347 |
| 26 | 40 | 3000 | 120 | 1867 | 0.370 | 0.386 | 0.490 |
| 27 | 40 | 3000 | 150 | 687.3 | 12.34 | 12.55 | 787.5 |
| 28 | 40 | 3000 | 180 | 224.5 | 226.5 | 267.2 | 396.0 |

time = 2.6 s) and $a_{t_{\min}} = -10\ g$ while coasting. The target is classified as a smart target.[4] This means that, when the missile gets to a relative distance of 6000 ft, the target rolls 45 deg and pulls 9 $g$ and that, when the time-to-intercept becomes 1.0 s, the target rolls 180 deg and pulls 9 $g$.

To apply the time-to-go predictors, the missile velocity vector is projected onto the plane of the target velocity vector and the line of sight. Then, the time-to-go is calculated in that plane and used for the three-dimensional engagement. For the time-to-go scheme based on minimum weighted final time (Sec. II), the constant missile velocity is taken to be the average missile velocity over the engagement time. This means that an iteration process is involved. The time-to-go is guessed; the average velocity of the missile is computed based on the approximate tangential acceleration mode; and the time-to-go is computed from Eq. (27). The process is iterated until the guessed time-to-go equals the computed time-to-go. If the predicted time-to-go is greater than 5.0 s, the time-to-go is set equal to 5.0 s. This is done for all time-to-go prediction schemes used here.

Results are presented in Table 1 where $n$ denotes the engagement number, $\theta_0$, $X_0$, and $\phi_0$ denote the off-boresight angle, the range, and the aspect angle at launch, and $d$ denotes the miss distance. The subscripts on $d$ denote range over closing speed (I), accelerated range over closing speed (II), minimum weighted final time (III), and minimum time with bounded control (IV). The specific formulas used for cases I and II are discussed briefly in the Appendix.

Table 1 shows that the time-to-go algorithm of range over closing speed (I) leads to misses ($d > 10$ ft) for most of the large off-boresight engagements, and this is the reason that the time-to-go algorithms presented in this paper ($\theta_0 \equiv$ given) have been investigated. As seen in the table, these algorithms (III and IV) give hits for the large off-boresight cases 22–26. Misses still occur for shots 15, 16, and 27, 28, but these are difficult shots, and misses occur for all time-to-go algorithms.

Of the two algorithms designed for large off-boresight angle, algorithm III gives slightly better but roughly the same results as algorithm IV. The surprise is that algorithm II, accelerated range over closing speed, gives miss distances very close to those of algorithm III. It is clear that, of the three algorithms II, III, and IV, algorithm II predicts the smallest time-to-go at each sample time. All three are accelerated, but the velocity vector in algorithm II is assumed to be directed along the straight-line intercept path. As a consequence, algorithm II leads to larger gains in the linear-quadratic guidance law and stronger maneuvering early in the trajectory.

These results seem to indicate that at each sample time there is a range of values of $t_{go}$ which give similar miss distance results. Furthermore, the values predicted by algorithms II–IV are within this range. In making a choice, algorithm II should be first on the list because it is much simpler.

## V. Discussion and Conclusions

Two minimum-time trajectories have been developed for use as a time-to-go algorithm for a missile controlled by a linear-quadratic guidance law. The first algorithm is based on the minimum normal-acceleration-weighted final time trajectory for a constant speed missile, and the second algorithm is based on the minimum final time trajectory of an accelerating missile with bounded normal acceleration. Because the initial off-boresight angle is fixed, these algorithms are expected to work well for intercepts where large off-boresight angles occur.

Both time-to-go algorithms are tested in a six-degree-of-freedom simulation of a bank-to-turn missile attacking a smart target, and their miss distances are compared with those obtained by time-to-go algorithms known as range over closing speed and accelerated range over closing speed. Both algorithms give low miss distances for a wide range of intercept geometries including those with large off-boresight angles. These results are substantially improved relative to those obtained with range over closing speed. However, the accelerated range over closing speed algorithm matches the minimum-time results and is the preferred algorithm because of its simplicity.

## Appendix

For a constant-speed missile and target moving in a homing triangle, the time until intercept or time-to-go is given by

$$t_{go} = (R/V_c) \tag{A1}$$

where $R$ is the relative distance along the line of sight and $V_c = -\dot{R}$ is the closing speed.

For a variable-speed missile, the tangential acceleration is assumed to satisfy the relations

$$t_{go} \le t \le t_c, \qquad a_t = a_{t_{\max}} > 0 \tag{A2a}$$

$$t_c \le t \le t_f, \qquad a_t = a_{t_{\min}} < 0 \tag{A2b}$$

where $t_c$ is the engine cutoff time. If the missile is assumed to be traveling at the average speed $\tilde{V}_M$ that accounts for missile tangential acceleration, the intercept condition

$$\tilde{V}_M \sin\theta = V_T \sin\phi \tag{A3}$$

leads to

$$\cos\theta = \left[\sqrt{(\tilde{V}_M^2 - V_T^2 \sin^2\phi)}/\tilde{V}_M\right] \tag{A4}$$

which defines the accelerated homing triangle. Then, the closing speed is given by

$$V_c = \tilde{V}_M \cos\theta = \sqrt{\tilde{V}_M^2 - V_T^2 \sin^2\phi} \tag{A5}$$

and Eq. (A1) gives the time-to-go. To complete the formulas, the average speed $\tilde{V}_M$ depends on the location of $t_0$ and $t_f$ relative to $t_c$. Hence, from

$$\tilde{V}_M = \frac{1}{t_f - t_0}\int_{t_0}^{t_f} V_M \, \mathrm{d}t \tag{A6}$$

the following relations are obtained:

$$t_0 \le t_f \le t_c, \qquad \tilde{V}_M = V_{M_0} + a_{t_{\max}} \frac{t_{go}}{2} \tag{A7a}$$

$$t_0 \le t_c \le t_f, \qquad \tilde{V}_M = V_{M_0} + a_{t_{\max}} \frac{t_{go}}{2} - \left[\frac{a_{t_{\max}} - a_{t_{\min}}}{2}\right] \frac{(t_{go} - t_c + t_0)^2}{t_{go}} \tag{A7b}$$

$$t_c \le t_0 \le t_f, \qquad \tilde{V}_M = V_{M_0} + a_{t_{\min}} \frac{t_{go}}{2} \tag{A7c}$$

where $t_{go} = t_f - t_0$.

The solution process is iterative for the time-to-go. First, a $t_{go}$ is guessed, and the average speed is calculated from Eq. (A7). Next, $t_{go}$ is calculated from Eqs. (A1) and (A5). Repeating the process using the computed $t_{go}$ as the guessed $t_{go}$ leads to convergence in a couple of iterations if the initial guess is close to the converged value. Hence, in calculating $t_{go}$ along an intercept trajectory, the $t_{go}$ from the previous update time is used as the initial guess during the current sample period.

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